

# How students structure their investigations and learn mathematics: insights from a long-term study<sup>☆</sup>

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## Abstract

This paper reports on the mathematical thinking of participants of a long-term study, now in its 17th year, who did mathematics together through their public school and early university years. In particular, it describes how fundamental ideas and images of a cohort group of students are elaborated and presented in symbolic expressions of generalized mathematical ideas while exploring problems in grades 10 and 11. From high school and university interview data, we learn from participants how they viewed their mathematical activity in structuring their investigations and justifying their solutions.

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## 1. Introduction

The main objective of this longitudinal research<sup>1</sup> has been to gain an understanding of how students think about mathematical ideas when they work together on well-defined investigations<sup>2</sup> with minimal outside intervention. This paper reports on a portion of the study, now in its seventeenth year. The

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<sup>2</sup> For sample tasks, visit [www.gse.rutgers.edu/rbdil](http://www.gse.rutgers.edu/rbdil).

mathematical reasoning of a focus group of students, who did mathematics together throughout their public-school years as participants in a research study, is described. The development of some mathematical ideas related to counting and ways of reasoning about Pascal's identity are described (Maher, 1998a, 1998b) by examining the conversations and inscriptions of a cohort group of five students. To illustrate their reasoning, two episodes from the study are presented, one from grade 10 and the other, from grade 11, in which the cohort group elaborated fundamental ideas and presented them in symbolic expressions of generalized mathematical ideas. Interview data from senior-high school and second-year university are included to show how participants viewed their mathematical activity in structuring their investigations and justifying their solutions.<sup>3</sup>

## 2. Background

This paper presents results from a larger study in which researchers posed open-ended and well-defined tasks to students, from four content strands,<sup>4</sup> in problem sessions that were not a part of regular-school mathematics instruction. The students were invited to work together, sometimes over long periods of time, and to present their ideas with suitable justifications that were convincing to them (and to us).

During their high school years, the same cohort met in informal after-school sessions (Maher, 2002, 2004). As university students, members of the cohort met individually and in small groups at the Robert B. Davis Institute for Learning.

## 3. A perspective on thinking and reasoning

The invitation to follow one's own level and search for various paths to investigate mathematical ideas is appealing. Freudenthal reminds us that there is no single path in the building of mathematical ideas (1991). His advocacy to provide learners with opportunities to find their own levels and to explore multiple paths challenges us to find situations to make this happen. How might we provoke thinking? What might learners do? According to Freudenthal (1991):

... the learner should reinvent mathematizing rather than mathematics; abstracting rather than abstractions; schematizing rather than schemes; formalizing rather than formulas; algorithmizing rather than algorithms; verbalizing rather than language. (p. 49)

Activity, thoughtfulness, and thinking have long been viewed as desirable human activities. Einstein (1936) referred to thinking as "operations with concepts". He described the thinking process as involving the creation and use of explicit functional relations between concepts and the coordination of sense experiences to these concepts. He wrote about the "awesomeness" of thinking and expressed scepticism about ever coming to understand how thinking actually works. Cognitive psychologists, cognitive scientists, and others interested in how learning occurs continue to study the process of thinking. Individuals and groups focus on different assumptions for their inquiry. The characterizations of processing systems, symbol manipulation paradigms, and the inferences made from observational and

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<sup>3</sup> For more detail, see Francisco (2004).

<sup>4</sup> The strands are algebra, counting and combinatorics, probability, and precalculus.

empirical research vary. The different frameworks for studying thinking and the assumptions that support these frameworks continue to be the subject of debate among theorists, policy makers, and practitioners. Because of the variety of frameworks and the controversy surrounding them, inquiry into the origin and growth of mathematical ideas and about how learners reason about their ideas is of special interest.

If, indeed, it is the responsibility of the learner to build up a collection of mathematical ideas in his or her mind, it is reasonable to ask where those ideas come from. Elsewhere, we have argued that new ideas come from old ideas (Davis & Maher, 1997; Maher & Davis, 1995). Learners have ideas in their mind and can work creatively with them. When they come to their investigations, they bring extensive personal experience. This experience, along with their existing mental representations, knowledge, and beliefs, may be modified and refined, as needed, in contexts that include a mixture of personal exploration and social interaction (Maher, 2002; Maher & Martino, 1992; Martino & Maher, 1999). Given a task, learners decide what to investigate, how to go about an investigation, and what to examine. In the process of their investigation, they compose a discourse of their own, and from this discourse, working theories emerge (Maher, 1998a, 1998b; Maher & Davis, 1995; Maher & Martino, 1997, 2001). What about the potential of working theories of learners? These theories can result in effective ways of working with mathematical ideas. Through personal experience with concrete objects, or with images derived from working with these objects, specific arguments are built (Maher & Speiser, 1997). When learners revisit their ideas from new perspectives, their arguments may serve as prototypes for later, more abstract reasoning. It is from learners' emerging theories and the way they work with them that an understanding of their work and thought can be built (Maher, Davis & Alston, 1992).

The theoretical foundation for the long-term study is grounded in research about learning mathematics, particularly research on representations. It is based on the perspective that in doing mathematics, mental images of mathematical ideas are formed by individuals (Davis, 1984; Davis & Maher, 1997, 1990; Maher & Davis, 1990, 1995). In the activity of problem solving, earlier images are retrieved, revisited, perhaps modified, and carried forth; new images are made. Through open discussion, as students explain, share, and argue about their ideas, certain features of their internal,<sup>5</sup> cognitive representations are made public (Dörfler, 2000; Maher & Martino, 1996a, 1996b; Maher & Speiser, 1997; Powell, 2003; Steencken, 2001; Uptegrove, 2005; Uptegrove & Maher, 2004a, 2004b). Usually, students cycle through explanations and arguments in an effort to produce valid justifications. When they are satisfied with their results, they may undertake a new investigation or propose and solve an extension to the original problem. In exploring new, self-initiated problems together, ideas are expressed more formally. Student discussions indicate that they notice connections among ideas built earlier and explore generalizations together (Kiczek & Maher, 1998; Kiczek, Maher, & Speiser, 2001; Powell, 2003; Powell & Maher, 2003; Uptegrove, 2005).

#### 4. Methods

The research presented here is an observational study detailing the collective building of mathematical ideas and ways of reasoning about the ideas by a focus group of learners. Thinking and reasoning are

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<sup>5</sup> Dörfler (2000) makes an important distinction between the visible representations and the internal, cognitive (re)presentations that are not visible to us.

documented by the actions of the participants, that is, what they do, say, and write. An extensive videotape database enables us to study the complexity of the group's learning.

We regard events as connected sequences of utterances and actions by the learners. An event is called critical when it demonstrates a significant advance from previous understanding or a conceptual leap in earlier understanding, also recorded as events (Kiczek, 2000; Maher & Martino, 1996a; Powell, 2003; Steencken, 2001). These episodes are obvious and striking in that they can be connected to prior events. Identification of critical events makes it possible for researchers to examine the influence of critical events on later understanding and to trace the development of ideas over time.

#### 4.1. *Data sources*

Our main sources of data are: (1) behaviours of students as they work on mathematical investigations recorded on videotape; (2) written work of students; (3) follow-up interviews of individual or small groups of students; (4) individual student interviews; and (5) researcher notes. Groups of researchers observed the students, took notes, noticed behaviours, and developed interests in what the students produced individually, together, and through sharing with others. At various times students revisited tasks and talked about their ideas. It was not uncommon for ideas to re-emerge for discussion over days, weeks, months, and years.

#### 4.2. *Videotaping*

Videotapes were made of nearly all sessions or task-based interviews. These videotapes have three main forms:

- (1) Videotapes of task-based (or “clinical”) interviews, where there would usually be one interviewer and one student being videotaped by two cameras (one to record work; the other to record the conversation.) Also, two or more observers took notes, but not in view of the student.
- (2) Videotapes (made outside the classroom in an office or quiet setting) similar to those just described, with two to five students working together on a task. During student investigations, there is usually no interviewer present. There are two cameras and a sound technician.
- (3) Videotapes made in actual classroom settings, but otherwise similar to the second category with small groups of students working together. There are three cameras and two sound technicians.

Results from our research emerge through systematic study of extensive, archived videotape data, often including older tape segments, which now, because of recent data, have been re-analysed from new directions with newly developed tools and a more detailed framework.<sup>6</sup>

#### 4.3. *Framework for analysis*

A framework is described that takes into account how ideas develop and travel within the group and how the teacher/researcher interacts in the process. The analysis begins with the identification of “critical events”. These events can be moments of mathematical insights or moments where cognitive obstacles surface. The mathematical content of each critical event is identified and described

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<sup>6</sup> See Davis, Maher, and Martino (1992) for a discussion of using videotapes to study the construction of knowledge.

taking into account the context in which the event appears, the identifiable student strategies and/or heuristics employed, earlier evidence for the origin of the idea, and subsequent mathematical developments that follow its emergence. These components, which we have called a trace, make it possible to track the development of idea(s) and are regarded as stories in students' development (Kiczek, 2000; Maher & Martino, 1996a, 1996b; Maher, Pantozzi, Martino, Steencken, & Deming, 1996; Steencken, 2001). We identify and code the *traces* in the form of diagrams. Concurrently, transcripts are verified and explicitly co-ordinated with diagrammed events. Our interpretations evolve from all of these.

The critical event itself defines the present. Prior images to which the critical event folds back define the past (both recent and more distant), while later events that help us understand (or fold back to) the present critical event define its future. This timeline is followed in strands of analysis, each of which is coded. Coded nodes denote events along the timeline, and descriptive codes are used to mark strands of events (flow of ideas).

#### 4.4. *Constructing a storyline*

The construction of a storyline begins with the flow of ideas. We examine and identify codes and their respective critical events in an attempt to trace an emerging and evolving story about the data. A storyline is constructed from a coherent organization of the critical events — a trace. This often involves complex flowcharting. Hence, the process of producing a trace involves identifying a collection of events, coding those events, and then interpreting them, to provide insight into a student's cognitive development. The trace contributes to the narrative of a student's personal intellectual history as well as to the collective history of a group of students who collaborate.

#### 4.5. *Writing a narrative*

In our model, a narrative phase enables researchers to view the recorded material from the data set holistically. Although it appears last, interpretative actions actually begin from the inception of research, originally formulated through theoretical perspectives and research questions of interest (Powell, Francisco, & Maher, 2003, 2004).

### 5. **Tasks and video segments**<sup>7</sup>

Part 1 illustrates how the cohort group of students worked together on their investigations. Two episodes from senior-high school, grades 10 and 11, respectively, are presented. In Episode 1, the group works on a problem posed by one of the members. In Episode 2, students explore Pascal's identity and explain, in general, how the addition rule works for Pascal's triangle. Part 2 gives excerpts from student interviews at the end of grade 11 and a few years later, as university students. These interviews provide insight into how students viewed their participation in the study.

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<sup>7</sup> See the Private Universe Project in Mathematics (PUP-Math) (2001) for a video clip of the episode and accompanying written materials.

## 6. Part 1: investigations

### 6.1. Episode 1: Ankur's challenge

In grade 10, five students (Ankur, Brian, Jeff, Romina, and Michael) met to consider variations of tower problems they had worked on in their early elementary years (Maher & Martino, 1996). They were asked to find how many 5-tall towers they could build, choosing from two colors, and to give a convincing argument for their solution. Ankur and Michael immediately found a solution and while they were waiting for the others to complete their argument, Ankur posed the following problem, introducing a third color and a particular condition:

*Ankur's Challenge:* Find as many towers as possible that are 4-cubes tall if you can select from three colors and there must be at least one of each color in each tower. Show that you have found all the possibilities.

The students partitioned themselves into two groups and worked on Ankur's new problem for approximately 15 min. Mike and Ankur, after calculating that there were 81 total towers when selecting from 3 colors, returned to the conditions of the problem and, by subtraction, came up with 39 towers. Romina, working with Jeff and Brian, said that there were 36 towers. Michael continued to work on the problem by himself. Unaware of the work of Romina and her group, he asks to hear their solution, indicating that he was now "ready to listen".

Romina went to the chalkboard and presented her justification. She indicated that the set of all possible towers could be partitioned into six groups. Since every tower would have two of one color, Romina focused on the placement of the duplicate color, using x's and o's. She indicated that for each placement of the first, or duplicate color, there would be two possible combinations for the second and third colors. She also indicated that these combinations would have two opposite arrangements for the second and third colors. She then tripled the 12 possibilities to represent every color, concluding that there should be a total of 36. See Fig. 1 for Romina's written work, presented at the after-school session, two weeks later.

The idea to use binary notation to count towers was introduced by Michael in an earlier session, in December 1997 (Muter, 1999; Muter & Maher, 1998). Other students soon integrated binary numbers in their coding for towers. Romina, in this episode, justifies her solution to Ankur's challenge, using ones and zeros to indicate the placement of the duplicate color. Romina shares the proof of the solution that she developed with Jeff and Brian, her partners. In her written work (Fig. 1), Romina used Y for yellow, B for blue, and R for the duplicate color. (Muter & Maher, 1998).

### 6.2. Episode 2: exploring Pascal's identity

In May of their junior year, the high school students returned to school one evening around 7:30 p.m. for a research session. The session began by asking the students to review what they had discussed in their pre-calculus class earlier that day. The class had touched on binomial expansions, and the students had learned to use a calculator to calculate the coefficient of any term without having to construct Pascal's triangle. The students were asked why the addition rule they used to build Pascal's triangle worked. In response, they showed a 1-1 correspondence between the terms in Pascal's triangle and particular pizza

## TOWER PROBLEM

HOW MANY TOWERS CAN YOU BUILD FOUR HIGH WITH THE CHOICE OF THREE COLORS AND HAVING ALL THREE COLORS IN EACH TOWER.

I first approached this problem realizing that if there are three different colors and four different areas where they can be put, then there will be a double of one color, and two distinct colors, in each tower. I then tried the problem with the colors yellow (Y), Blue (B) and Red (R) with red being the duplicated color:

R	R	Y	B
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B	R	R	Y
---	---	---	---

R	Y	R	B
---	---	---	---

B	R	Y	R
---	---	---	---

R	B	Y	R
---	---	---	---

B	Y	R	R
---	---	---	---

6  $\leftarrow$  positions of color being dupl  
 $\times 2$   $\leftarrow$  combo - BY or YB  
 12  $\leftarrow$  total for red being dupl  
 $\times 3$   $\leftarrow$  # of colors  
 36  $\leftarrow$  # of combinations

I placed the two reds in every possible way they could be, which left me with 6 possibilities. In the remaining blocks there can only be a combination of two possibilities: yellow and blue or blue and yellow. Since there are only two remaining possibilities of combinations one must multiply the 6 by 2 becoming 12. The 12 represents the number of possibilities of towers being built with R as the duplicated color. Since there are 3 colors, and each can be the doubled color, one must multiply

the 12 by 3, indicating the red, yellow, and blue have been the duplicated color in each. Leaving one with the answer of 36; the number of all possible combinations.

Fig. 1. Romina's written work of her solution to Ankur's challenge.

$$\binom{N}{X} + \binom{N}{X+1} = \binom{N+1}{X+1}$$

Fig. 2. Students' board work of Pascal's identity.

$$\left( \frac{n!}{(n-x)!x!} \right) + \left( \frac{n!}{(n-x+1)!(x+1)!} \right) = \left( \frac{(n+1)!}{(n-x)!(x+1)!} \right)$$

Fig. 3. Students' boardwork with factorials for Pascal's identity.

and tower problems. In this way they gave meaning to the addition rule in their discussion of these problems.<sup>8</sup>

The students wrote Pascal's triangle on the board using the notation  $\binom{n}{r}$  to denote the number of combinations of  $n$  things taken  $r$  at a time. They wrote the general  $n$ th row in Pascal's triangle of  $n$  things taken  $x$  at a time. The students responded to the researcher's request of formulating the addition rule with this notation. They explained the correctness of the notation by referring, first, to particular cases of the pizza problem<sup>9</sup> and then to the meaning and structure of the addition rule as additional toppings are added.<sup>10</sup>

They wrote the equation shown in Fig. 2.

Challenged by the researcher to express their result in factorial notation, the students worked together to produce the equation shown in Fig. 3.

When asked by the researcher to examine their boardwork, they rewrote the denominator of second term to read  $(n - (x + 1))! (x + 1)!$

After successfully writing the addition rule for  $\binom{n}{x}$ , Jeff remarked: "Do you know, like, how intimidating this equation must be like if you just picked up a book and looked at that".

Mike, in an interview 3 years later (April 2002), was again asked how he might explain the addition rule. Mike indicated that he remembered that the group had given a general rule in the 1999 after-school session and began, immediately, to reconstruct it. This time, Mike called a row  $r$  [to denote the number of toppings], and a "spot" in the row,  $n$ . He used the notation " $r$  choose  $n$ " + " $r$  choose  $n + 1$ " to show " $r + 1$  choose  $n + 1$ ", referring again to adding toppings to a pizza to explain the rule. In the same interview, Mike talked about how he looks for relationships while he works on problems and said:

The process while I am doing the problem . . . I just start understanding more that this is related to that, how this is related to just a triangle that's made up of numbers. At first when they showed us a

<sup>8</sup> For a detailed discussion, see Muter (2000) and Uptegrove (2005).

<sup>9</sup> The metaphoric reference is to the general pizza problem. Since grade 5, the students worked on variations of pizza investigations. The reference here is to a general problem of finding how many different pizzas that could be made using any number from, say  $n$ , different topping choices and of considering how they could account for organizations of pizzas as additional toppings are made available.

<sup>10</sup> For a full discussion, see Kiczek (2000), Kiczek and Maher (1998), and Kiczek, Maher, and Speiser (2001).



triangle, we didn't know that has anything to do with . . . once you start understanding things have a relation to each other you just start convincing yourself . . . and then you come to a point where you know it's right, or you think it's right.

The mathematical reasoning displayed by the students suggests their deep understanding of basic ideas and their thoughtfulness when reasoning together. In Part 2 below and in the work by Francisco (2004), we again see how deeply some participants viewed their activity and how they understood that they were structuring their investigations over time.

## 7. Part 2: reflections by students on their participation

Through a series of individual and small group interviews, students' views are offered on how they viewed participation in the long-term study.

### 7.1. *We'd get together and present our ideas*

In a May 1999 interview, 17-year-old Jeff describes the way they worked and how ideas were dealt with:

Well, we break up into groups . . . like five groups of three, say, and everyone in their own groups would have their own ideas, and you'd argue within your own group, about what you knew, what I thought the answer was, what you thought the answer was and then from there, we'd all get together and present our ideas, and then this group would argue with this group about who was right with this . . .

The notion of argument was key to the way the students worked together. For example, to represent an idea, an individual may create a structure or present a notation. In this way, the ideas are made public in their discourse in the form of explanations, actions, writings, and notations. Records make possible later re-examination of the relationships between ideas. In this way, ideas presented earlier would again be visited, discussed, and thought about. This process was entirely student driven and made possible by individual initiatives.

### 7.2. *We were just trying to do our own thing*

In the same interview, Jeff also talks about student ownership of the mathematics they were building:

You didn't come in and say this is what we were learning today and this is how you're going to figure out the problem. We were figuring out how we were going to figure out the problem. We weren't attaching names to that but we could see the commonness between what we were working on there, and maybe what we had done in school at some point in time and been able to put those things together and come up with stuff and to do these problems to come up with, what would be our own formulas because we didn't know that other people had done them before. We were just kind of doing our own thing, trying to come up with an answer that was legitimate and that no matter how you tried to attack it, we could still answer it. It was a solid formula that works no matter how you tried to do it.

### 7.3. *All we did was just explore*

Mike (May 1999) talks about exploring, days at a time, and gaining understanding:

In our class, all we did was just explore. We took days at a time, and I have a good understanding of it . . . like, if you were going to, I guess, a normal class, you'd have to be, like, only selected kids might understand it. But in a class where everybody's working together, everybody's a part of the teaching, and everybody, or at least the majority of kids will understand it.

### 7.4. *I would like to work on it before I came up with a couple of options myself*

Romina (March 2002) comments on the way they worked:

If I didn't understand the problem, or if I didn't work enough to it, by myself to understand, and I guess if Michael didn't know where I was heading with what I was doing, and if I didn't understand where the other person was heading I would like to work on it before I came up with a couple of options myself to see which one we take.

### 7.5. *Just giving the answer was not enough*

Jeff (May 1999) indicates that the students, themselves, took on the expectation for presenting a careful argument. Consequently, they reviewed their own argument, focused on meaning, and anticipated questions. They questioned each other and put ideas together before offering their solution to the researchers. Listening to, and asking questions of, each other were essential components of the process of working together.

Just giving the answer was never enough, in order to do it. You'd have to have a good, like, structural record. It's almost like doing, like a proof . . . like you need to show every step from point A to point B . . . you couldn't just, like, skip some things and jump around. You had to go straight, and everything had to be written out and good, and [with] understanding, and if you had a problem with somebody, to ask another question about it, so you ended up doing whole types of things, just to get from the beginning to the end, and through it, that's how you really understand what you were doing, that's why we'd learn, like what we were doing without actually calling anything a certain thing, you know.

### 7.6. *We would come to it ourselves*

Jeff (May 1999) talks about not being told how to do things:

And then, like, later now, we would be doing things, like, "Oh, that's what we were learning". Because Rutgers never really told us what the answers were, or what we were actually doing . . . like, 'This is what we're going to do today; it's called the . . . theorem,' or anything like that. We would come to it ourselves, and then later, we would realize that that's what we were doing this whole time.

### 7.7. *We'd really have to start from bare bones*

Jeff (March 2002), age 19, talks about starting from “bare bones” and how they did not talk to the researchers until “very late in what we were doing”:

If we tried to just present a final thing and really didn't know it from the beginning we couldn't explain it in a way that that you would accept from us. So in order to explain it in a way that you would accept we'd really have to start from bare bones, from the beginning . . . We didn't start talking about what we were doing with you until very late in what we were doing. There was not a lot of communication back from them to us about the work we were doing.

### 7.8. *We got so in-depth*

Jeff (May 1999) reviews what they accomplished:

Well, even though we didn't spend much time together, and they [researchers] only came a few times a year, we did so much, we covered so much, we got so in-depth on topics, that it leaves an impression. I mean, we could talk about doing the blocks in first grade, and we can almost go through problems: We did shirts and pants in second grade. I mean, how many other people can tell you the math that they were doing in second grade . . . like a word problem, you know? Because you go in deep, you work on it so much, and you go so far into it, that it just sticks with you . . . That's why it leaves such an impression, because of the depth you get into it, you know.

### 7.9. *We just sat and thought for hours a day*

Romina (July 1999), age 17, talks about the confidence she gained. She indicates that they spent days thinking about problems and further that presenting them to others was a valuable undertaking.

We did a lot of problem solving. We did a lot of thinking. We just sat and thought for hours a day, and we came up with a lot of interesting things. We were able to go in front of a large audience and talk about our ideas and argue our points, and prove our points. I think it was a very good experience.

### 7.10. *I think a lot of what we were doing was working together*

Jeff (February and March 2002, respectively) talks about the benefits of collaboration and the frustration of working alone.

Well that's . . . how we got to wherever we were going . . . we were like four different people four ideas and we all thought we knew something on how to do a problem but . . . you just cover so much more when everyone is discussing what you're doing, I mean that's what it was really all about . . . that's really how we got anywhere was kinda work, doing our thing together, you know, and using what we each knew, to work something out.

I think it would have been very different if it was all of us producing our own solutions . . . I think a lot of what we were doing was working together. I think when you are working alone, when you reach a part where you don't know anymore [that] it is very easy to just be frustrated and say I

don't know anymore. I'm not going to do this. I can't think about this. Like forget it. I think that by working with everybody when you got to that point, you can kind of peak over a little bit and it was all right . . . it was encouraged. That allowed everybody to really we could all move forward.

### 7.11. *Everything has to have Romina's definition*

Romina (March 2002) talks about her need for sense-making.

Everything has to make sense in my terms. Someone else may have done it already in a book, but I just don't understand it unless I try it myself and put it in my own terms.

The need for sense — making, working together, negotiating reasonableness, and personal ownership of ideas characterizes the views of these students on how they learn mathematics. There is hardly dispute about the depth of their understanding. What are noteworthy are their articulate expressions of how their personal understanding grew.

## 8. Conclusions and implications

The wealth of knowledge gained from the study extends well beyond researchers' original expectations, that is, to learn about how mathematical ideas are built by learners when certain conditions were in place. While, on the surface, the implementation of those conditions might appear to eliminate "teaching", it must be recognized that, by design, minimal intervention by researchers was the intent. Our attention was to focus on knowing how certain knowledge was built by students who were invited to *do* mathematics. By shifting the responsibility to the learners, they responded. They were driven by a desire to make sense of their mathematical activity. In fact, the importance placed by the students, themselves, to make sense of their activity led to careful reasoning and building of arguments. The culture of sense — making that emerged early in the study led to arguments and justifications. Their discourse, naturally, involved arguing about ideas and providing convincing evidence to each other. This, in turn, led to their proof making and generalizing. The processes developed in students through their activity in doing mathematics in the context of coherent strands of investigations that they were invited to explore.

Reflections by students about their learning over the project years give insight into how they viewed their work together. Jeff reported that giving an answer was never enough. He indicated that they would be expected to provide a written account to support their reasoning and that details in arguments were important. They accepted that they would not be told answers or how to solve a problem and very early they took on the expectation that they would produce a result and offer appropriate support for it. The support came first from convincing themselves and then from convincing each other. Their own expectations guided how far they were willing to go in solving problems. They were attentive to details; they expected to provide evidence for their results; and they were ready to consider extensions and generalisations to investigations posed.

There was a keen awareness by the students that they, themselves, were responsible for the doing of mathematics. As evidenced by their comments, they took on, progressively, responsibility for their own learning and for the maintenance of communication and collaboration within their working groups. They maintained high expectations for themselves and for each other. Indeed, these expectations may help explain the way they worked together and their commitment to the project over so long a time.

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