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3. VIDEO RECORDINGS AS PEDAGOGICAL TOOLS IN MATHEMATICS TEACHER EDUCATION

Teachers do not ordinarily have the opportunity to closely observe students' learning mathematics in their own classrooms. Studying well-chosen video recordings can serve as a seminal resource for understanding how students represent mathematical ideas and reason about them. Analyzing videos of students engaged in doing mathematics can be helpful to teachers in recognizing how mathematical ideas develop in learners as well as how valid justifications for solutions to problems are built by learners. Video episodes that illustrate various forms of student mathematical reasoning can help teachers become better aware of the unrealized potential of their own students' thinking.

This chapter focuses on the use of videos in teaching. It is organized in five sections. The first section provides an overview and rationale for using videos as pedagogical tools in studying learning and teaching - both from the perspective of using videos from collections and by making new videos to study one's own practice. In the next two sections, examples are provided of how video collections have been used as pedagogical tools for teacher education and professional development and for the study of learning and teaching. The next section gives examples of the use of an extensive video collection at Rutgers University for teacher education and provides an example of a secondary school teacher's study of videos of children's reasoning. In the final section future directions in the collaborative use of videos for teacher education are suggested and a summary for the use of videos as a tool for teaching is offered.

INTRODUCTION

Video recordings can be effective pedagogical tools for enhancing teacher learning in mathematics teacher education. Whether one studies the videos of one's own teaching or the teaching of others, video recordings invite conversations about student learning and teacher actions. They can illustrate a variety of classroom conditions for learning and teaching. In addition, video collections containing extensive sets of classroom teaching and learning are influential in mathematics teacher development.

Since videos can capture aspects of the emerging processes of learning, they can serve to engage prospective and practising teachers in new strategies for effectively teaching mathematical concepts to a range of students. Another significant feature in the use of videos for improving teacher practice is that videos make possible the study of teaching moves that play an important role in influencing learning outcomes. Teacher educators and prospective and practising

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Video Study of One's Own Teaching

Video recordings are useful in studying one's own teaching. Teachers can evaluate whether or not certain interventions were beneficial and timely for student learning and reflect upon their practice. By studying video recordings of their own teaching, certain interventions can be reviewed and their consequences considered. Teacher moves can become the object of public discussion in which consideration of alternative approaches can be proposed and, under similar conditions, later tried. Depending on one's goals, video recordings are powerful tools for the detailed study of learning and teaching.

Teachers, after reviewing their teaching through the study of classroom video data, can become more aware of their practice. An example of the use of video recordings for this purpose is *video clubs* as employed by Sherin (2007). She used a video club format to study the professional development of a group of middle-school mathematics teachers who met monthly over a year to share segments of video recordings of their lessons. Together, the teachers observed excerpts of their video recordings. Prompted by a facilitator who asked them about what they noticed, the teachers responded by drawing attention to certain events. According to Sherin, this format evoked discussion and reasoning about certain events. She reported that teachers developed new ways of reasoning about student conceptions, thus showing development in the teachers' professional vision of students' mathematical understanding.

Using videos as a tool to evaluate and improve one's practice, teachers can plan and implement lessons, study the videos of the lessons, and analyze students' developing understanding as they take into account their role as facilitators in the process. Teachers can also reflect on the extent and quality of their probing of student understanding (Martino & Maher, 1999). Video recordings of lessons can capture the explanations teachers give to students and the responses they offer to student questions. In this way, teachers can observe and learn from their own practice and the practice of others in their community. By studying their lessons, teachers can become more aware of their classroom behavior. They can reflect on the moves they make and then consider and discuss with others whether or not certain moves are effective (Alston, Potari, & Myrtil, 2005).

In comparison with other evidence of students' learning, such as verbal statements and explanations, written work, and other forms of evaluation, video recordings offer the opportunity for studying subtle details of students' evolving learning. Obstacles and creativity in individual learning can be captured for later examination and review.

Video Portfolios

Another advantage of using video recordings to study one's practice is the potential for building *video portfolios* of student learning and teaching for later study. Included in video portfolios might be the collection of different kinds of data centered at an episode or a series of episodes of interest. These data sources may include: (a) video "cuts" of critical events of student learning; (b) video "cuts" of critical teaching episodes; (c) associated written work of students; and (d) teacher and researcher notes documenting the mathematical activity that provide a 'trace' or pathway of the development of a mathematical idea (Maher & Martino, 1996a). An example of the use of video data to trace a student's cognitive evolution from pattern recognition to theory posing is contained in Maher and Martino (2000).

VIDEO COLLECTIONS OF MATHEMATICS TEACHING AND LEARNING

Collections of video recordings provide a database for careful analysis and reflection on student learning and on pedagogical practice. As these resources become increasingly more available through website access, links to storage areas and public repositories, mathematics teacher educators and professional developers can incorporate their use, taking into account language, cultural and context differences, into learning opportunities for teachers. A valuable feature is their availability for multiple viewings of consecutive lessons. Revisiting episodes of classroom lessons makes possible group discussion and personal reflection on one's own practice. It provides access to a variety of teaching styles in orchestrating classroom environments and the ways in which students respond to these. Multiple viewings provide for in depth analysis and reflection on the developing ideas and growth in conceptual knowledge of learners, as well as the obstacles they encounter in their learning. In this section, examples are given of projects that use video collections as tools for teacher education and professional development.

VIDEO CASES

A review of the use of video cases in general teacher education and mathematics teacher education specifically suggests that these videos are at least as effective as other methods (including live observation) in teacher training and are more effective than other approaches in helping prospective teachers learn teaching strategies and content (Grossman, 2005; Philipp, et al., 2007) (see also Seago, Volume 3, for a discussion of video cases as a means for practising teachers to develop as professionals). Video cases, specifically, make use of more advanced technology such as hypermedia in order to link certain video recordings to other artifacts. Examples of the use of video cases for teacher education and the study of students' learning are Derry and Hmelo-Silver's STELLAR System, Lampert and Ball's University of Michigan 'library collection', and Maher et al.'s longitudinal video cases described in the Private Universe Project in Mathematics (PUP-Math).

The STELLAR System

An example of an arrangement using video cases is STELLAR, a system that contains a collection of 11 video cases that are approximately 25 minutes long and indexed to a learning sciences hypermedia. The cases were collected primarily for use in learning sciences courses with students from multiple disciplines, including mathematics teacher education. Students using the system can examine which learning sciences concepts (e.g., transfer, cognitive apprenticeship) might be relevant to a particular video case. Also, for each learning sciences concept, students can link to all the video cases that exemplify that concept. These video cases were used as part of a problem-based learning (PBL) approach to instruction, in which the video cases are used as contexts for instructional design problems. A study of learning outcomes for prospective teachers in this PBL course demonstrated that students using video cases integrated with the online STELLAR system developed deeper understanding of targeted learning sciences concepts than students in a comparison class (Derry, Hmelo-Silver, Nagarajan, Chernobilsky, & Beitzel, 2006).

University of Michigan 'Library Collection'

Lampert and Ball conducted pioneering work in using video cases for teacher preparation and professional development by creating a video library in the domain of mathematics education. The library is a set of hypermedia materials based on their own teaching for an entire school year of instruction in third and fifth grades. The National Council of Teaching Mathematics (2000) *Principles and standards for school mathematics* were included in the hypermedia materials but, because of technical limitations at the time of development, the video and NCTM (2000) Standards are not cross-referenced (Lampert & Ball, 1998).

Much of the video used for professional development at the University of Michigan was collected originally for research purposes by Lampert and Ball to support the study of their 'reform-based' teaching. The resulting materials for use with teachers contain not only videos cases but also additional artifacts, transcripts, records of students' work, and their teacher journals. Using these video cases as a tool in teacher preparation, the prospective teachers were able to investigate how student ideas developed over the course of a year and how the teacher/researchers (Lampert & Ball, 1998) facilitated the process. An outcome of the project was a database for the sequence of classes, some of which were edited because of storage limitations. The organization of the collection was by day and unit of instruction, as compared, for example, with organization by mathematical concepts.

Private Universe Project in Mathematics

The Private Universe Project in Mathematics (PUP Math), *"Problems and Possibilities"*, is a video workshop for K-12 educators investigating how mathematics teaching can be structured to resonate with children's own mathematical ideas – many of which are surprisingly complex. The video cases of children's reasoning drawn for the video workshops come from the collection at the Robert B. Davis Institute of Learning, Rutgers University. Accompanying the six, one-hour video programs are materials for users (Maher, Alston, Dann, & Steencken, 2000). The Annenberg Channel broadcasts PUP-Math materials to schools, colleges, libraries, public broadcasting stations, public access channels,

grade mathematical ideas and insights to their justifications and building of proofs in later grades. Drawing from this data and parallel studies in school and informal, after-school settings, a collection of over 3500 hours of digitized video produced an extensive video library. Included in the collection are transcripts, tasks (organized by strand), student work, and researcher notes (Francisco & Maher, 2005). The various mathematical content areas addressed in the study emphasized counting and combinatorics, early algebra, probability, and pre-calculus, with a focus on the development of reasoning by student learners. The digitized video data are now housed at the Robert B. Davis Institute for Learning (RBDIL), Rutgers University.

VIDEO COLLECTION FOR TEACHER EDUCATION

The video collection at Rutgers University (described above) contains unique and valuable video data and artifacts such as transcripts of video recordings, coding schemes, students' work, observational and research notes, analytic commentaries) on how students build mathematical ideas and ways of reasoning over time in a variety of diverse school settings and across all grade levels, in several content domains. Drawing from the longitudinal and parallel cross-sectional studies in urban and suburban contexts, a collection of videos has evolved that captures the development of children's reasoning, early proof making, and how students build isomorphism between and among problems of the same structure (Maher, 2002, 2005; Maher, Muter & Kiczek, 2006; Maher & Martino, 1998; Maher & Martino, 1996; Maher & Davis, 1996; Maher, Davis, & Alston, 1992). This collection provides a useful resource for interdisciplinary teaching and research (Maher, 2005; Powell, Francisco, & Maher, 2003, 2004; Speiser, Walter & Maher, 2003; Maher & Speiser, 1997). The collection has the potential to be extraordinarily useful for mathematics teacher education.

The knowledge required to teach mathematics from the perspective of developing thoughtfulness and meaning in student learning requires that teachers have a deep understanding of the mathematics they are expected to teach and knowledge of the cognitive development of children in those areas. A model for use with prospective and practising teachers that has evolved using video recordings from the Rutgers archive involves: (1) teachers studying mathematics by working on strands of tasks; (2) teachers collectively studying their own solutions; (3) teachers viewing and analyzing video recordings of children working on the same or similar tasks; and, (4) teachers implementing and analyzing, together, the same or similar lessons in their own classrooms. In both undergraduate and graduate courses at Rutgers University, the videos are used to supplement classroom investigations, readings, and discussions. Prior to studying video data, prospective and practising teachers engaging in doing mathematics together, sharing their solutions to the problems, and learning about the contextual background and setting of the video episode they will be viewing. An example of this process is provided in the section that follows.

Doing Mathematics: Building Towers and Making Pizzas

As a component of teacher development, participants worked together on the following investigations: Building Towers and Making Pizzas. Following their collaborative work, teachers were invited to share their solutions to the problems. These problems were:

> Building Towers: *Build all possible towers four cubes tall when two colors of unfix cubes are available; then provide a convincing argument that all possible arrangements have been found.*
>
> Making Pizzas: *A local pizza shop has asked us to help design a form to keep track of certain pizza choices. They offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms, and pepperoni. How many different choices for pizza does a customer have? List all the possible choices. Find a way to convince each other that you have accounted for all possible choices.*

Studying Videos of Children Doing Mathematics

Following the activity of working on the tasks and sharing solutions, the teachers work together to study and analyze pre-selected videos episodes. As an example, consider the video, *Brandon and the Pizza Problem*, from Workshop 3, of PUP Math, in which nine-year-old Brandon explains the isomorphism between the 4-tall Tower Problem and the 4-topping Pizza Problem (Greer & Harel, 1998). See www.learner.org/channel/workshops/pupmath, Workshop 3.

THE CASE OF MANJIT, A HIGH-SCHOOL MATHEMATICS TEACHER

Manjit is in her seventh year of teaching in a suburban high school near Rutgers University. Over the years, she taught a variety of classes from remedial to honors-level mathematics. Last year, she began participating in off-campus courses as a component of a district-wide professional development program that emerged as a partnership between her school district and the University. She now is pursuing a graduate degree while teaching.

As a component of a counting/combinatorics strand on mathematical reasoning and justification (Workshops 1-3, PUP-Math), Manjit built solutions to the Tower [illegible] viewed several videos from the PUP-Math [illegible]

Manjit's Description of Solutions to Pizza and Tower Problems

Manjit worked in class with her group on the 4-tall tower and 4-topping pizza problems. She reports below on her solution to each of the problems. She wrote:

> *When I was assigned the Pizza and Towers problems, I organized my information in a table each time. For the pizza problem I found all the possible pizzas that can be made choosing from four different toppings by grouping the pizzas by the number of toppings each pizza had. The categories I used were pizzas with 0 toppings, 1 topping, 2 toppings, 3 toppings, and 4 toppings. When I started working out the problem I realized that I needed to account for 16 pizzas due to the binomial nature of the problem (because each topping can either be on or off and there are 4 toppings thus total number of pizzas equals $2^4=16$) I was able to convince my group that I had found all the pizzas by organizing all the cases by groups as shown in the table below.* (See Figure 1)

Possible Pizzas	Pepperoni	Mushroom	Sausage	Onion	TOTAL
0 Toppings					**1 pizza**
1 Toppings	**X**				
		X			
			X		
				X	**4 pizzas**
2 Toppings	**X**	**X**			
	X		**X**		
	X			**X**	
		X	**X**		
		X		**X**	
			X	**X**	**6 pizzas**
3 Toppings	**X**	**X**	**X**		
	X	**X**		**X**	
	X		**X**	**X**	
		X	**X**	**X**	**4 pizzas**
4 Toppings	**X**	**X**	**X**	**X**	**1 pizza**

Figure 1. Manjit's solution for the 4-topping pizza problem

When I was assigned the problem to find all towers possible that were 4 tall choosing from blocks of two colors, I again accounted for all the towers in a tabular format by controlling how many red blocks were used in a given tower. As I was making the table, (see Figure 2), I realized that the towers with one red block can also be thought of as towers with 3 blue blocks.

Group	1st place	2nd place	3rd place	4th place	Group	Total
0 Red					All Blue	1
1 Red	X				3 Blue	
		X				
			X			
				X		4
2 Red	X	X			2 Blue	
	X		X			
	X			X		
		X	X			
		X		X		
			X	X		6
3 Red	X	X	X		1 Blue	
	X	X		X		
	X		X	X		
		X	X	X		4
All Red	X	X	X	X	0 Blue	1
						16

Figure 2. Manjit's solution of the 4-tall tower problem.

Manjit's Recognition of the Isomorphism to Pascal's Triangle

In describing the relationship of the tower and pizza problems and their relationship to Pascal's triangle, Manjit wrote:

As I finished making the table, I was able to see the fourth row of the Pascal's Triangle in the total number of towers for each category just as I was able to see this same row in the pizza problem solution. The solution for the towers problem is shown in the table (see Figure 3) below.

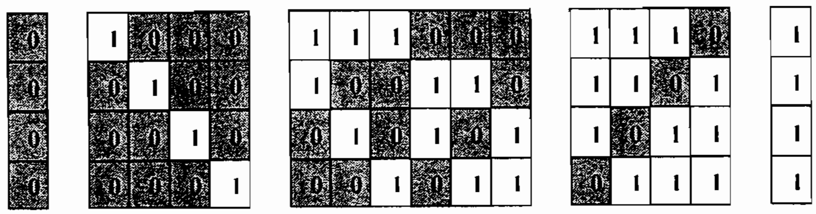
					1						Row 0
				1		1					Row 1
			1		2		1				Row 2
		1		3		3		1			Row 3
	1		4		6		4		1		Row 4
1		5		10		10		5		1	Row 5

Figure 3. Manjit's representation of Pascal's Triangle.

Manjit's Analysis of the video of Brandon's Problem Solving

After studying the videos of Brandon, a ten-year old fourth-grade student, and his interview about his solution to the 4-tall pizza problem, Manjit remarks on Brandon's *aha*, and his recognition of the isomorphism between the two problems.

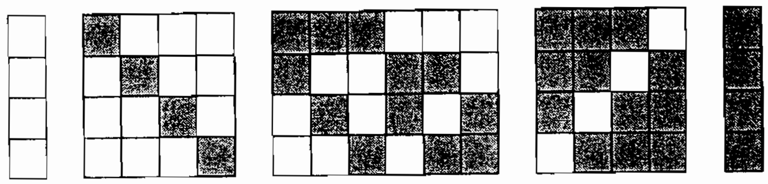
> *After we presented our solutions to the class we had an opportunity to watch Brandon's explanation of his solution for the Pizza problem, which involved coming up with all possible pizzas if you had four toppings available. In this clip Brandon explains how he made his table by number of toppings on each pizza. He uses a "1" to show the presence of a particular topping and a "0" to show its absence. Figure 3 shows Brandon's table that he made in the class. A month later Brandon was interviewed about his solution for the pizza problem and he recreates the pizzas by groups, which are controlled by the number of toppings in each group. He methodically lists his pizzas by zero toppings group which only has one possibility which he records by putting a "0" under every topping heading to show that topping's absence. For his one topping group he starts off with assigning a "1" to the peppers column and making the remaining three columns "0". He then moves to the mushroom column and puts a "1" under that column and "0" in the other positions. He continues this pattern until he has a "1" under each column as shown in Figure 4. For the two toppings group Brandon again uses a very systematic way of recording his pizzas. He starts off with "1" under the first two columns and "0" in the next two columns. For the next pizza in this category, he keeps the first column constant and now puts a "1" under the third column and puts a "1" in the last column. Then he puts "1" under both the second and third column and "0" in the other spots. Next row he keeps the "1" under the second column and now pairs it with the last column. Using this scheme he is able to account for all possible pizzas.*



Brandon's organization of the possible pizzas	P Corresponds to the top block	M Corresponds to the second block in the tower	S Corresponds to the second block in the tower	Pepperoni Corresponds to the second block in the tower	Number of pizzas in each category
0 toppings	0	0	0	0	1 pizza
1 topping	1	0	0	0	
	0	1	0	0	
	0	0	1	0	
	0	0	0	1	4 pizzas
2 toppings	1	1	0	0	
	1	0	1	0	
	1	0	0	1	
	0	1	1	0	
	0	1	0	1	
	0	0	1	1	6 pizzas
3 toppings	1	1	1	0	
	1	1	0	1	
	1	0	1	1	
	0	1	1	1	4 pizzas
4 toppings	1	1	1	1	1 pizza

Figure 7. Manjit's rendition of Brandon's isomorphism between pizzas and towers

He goes even further (see Figure 8) by explaining that the problem could have been solved even if we assigned the value of "1" to the red blocks instead of the yellows (see Figure 8). He justifies this statement by rearranging his towers in the following fashion.



Brandon's organization of the possible pizzas	P Corresponds to the top block	M Corresponds to the second block in the tower	S Corresponds to the second block in the tower	Pepperoni Corresponds to the second block in the tower	Number of pizzas in each category
0 toppings	0	0	0	0	1 pizza
1 topping	1	0	0	0	
	0	1	0	0	
	0	0	1	0	
	0	0	0	1	4 pizzas
2 toppings	1	1	0	0	
	1	0	1	0	
	1	0	0	1	
	0	1	1	0	
	0	1	0	1	
	0	0	1	1	6 pizzas
3 toppings	1	1	1	0	
	1	1	0	1	
	1	0	1	1	
	0	1	1	1	4 pizzas
4 toppings	1	1	1	1	1 pizza

Figure 8. Manjit's rendition of Brandon's recognition of the equivalence of colours.

Manjit's Study of Video Data

In her analysis of the video of Brandon's reasoning, Manjit pointed out his use of notation, his organization by cases, and his recognition of the structural equivalence of both Pizza and Tower problems. Her detailed analysis of the video data enable her to recognize the depth of Brandon's problem solving, recognize the isomorphism he built, and relate her representation of the solution by cases to a row in Pascal's triangle. She writes as follows:

> *This explanation by Brandon for the Pizza and the Towers problems is a stellar example of an isomorphism that was invented by the student. This shows an extremely deep level of understanding of these two problems. I have watched many other students come up with brilliant explanation for*

their answers and they are able to justify their answers which in my opinion builds a strong bases for life long learning for these students.

It is noteworthy that the problem solving of a nine-year old student triggered such rich mathematical thinking from a secondary school teacher. By studying the representations, heuristics and ways of reasoning displayed on video recordings of young children, teachers and those preparing to be teachers, can, like Manjit, also think deeply about the rich mathematical ideas stimulated not only by Brandon's problem solving but their own problem solving as well.

SUMMARY AND FUTURE DIRECTIONS

Studying video cases and video data as a single video, video cases, or from large-scale collections can assist teachers to grow in their pedagogical content knowledge. The videos have invaluable potential for developing an awareness of how students use existing ideas and build new ones. Video research data can suggest new instructional strategies and ways of working with children and assist those who are learning to teach and those who want to improve their teaching. Certain teaching and learning episodes can be helpful to visualize abstract ideas, and provide access to phenomena and evidence that can help influence learning (Davis, Maher, & Martino, 1992; Derry, Wilsman, & Hackbarth, in press; Maher, 1998; Powell, Francisco, & Maher, 2003, 2004; Philipp et al., 2007).

While large-scale projects often collect large amounts of video data, often only a small subset of these data is relevant for addressing particular questions of learning, teaching and other areas of interest. Video repositories can store video that was carefully selected in a systematic way and illustrate particular critical events in learning and teaching. The events, when selected and edited, tell a story or make a point. A future vision for video use is for teachers and researchers to have access to video stored across multiple distributed servers so that they can contribute their interpretations and analyses of the videos. An outgrowth is to establish communities of collaborators who, together, have the potential to advance knowledge about learning and teaching (Goldman, 2007; MacWhinney, 2007; Pea and Hoffert, 2007). The Robert B. Davis Institute at Rutgers is working to build such a collaboratory.

A decade ago, Goldman-Segall challenged us to continue to grow through active engagement in building new meanings as we study videos and share our interpretations with others. Still worthy of reflection today is the notion that:

The learner of the future is a constructor of knowledge, a meaning-maker; the digital culture has made us more aware of our being meaning-makers. We are not only beholders and readers, we are creators. We create our truths, our meanings, through our points of viewing; and we do this while making new meanings both for ourselves and for other. However, we cannot make meaning from the plethora of choices we are offered each moment in an electronic environment if we are passive about exploring our own and others' point of viewing. The purpose of making our perspective clear is to invite others to explore how we see the world. In other words, our goal is to create *a platform encouraging full participation in this wondrous exploration. We do not end our search by understanding our own perspectives; that is where we begin our journey. We build out from the center to fringes that may, in turn, become new centers. And we continue learning by looking for new perspectives that deepen and broaden our ways of looking at the world around us.* (1998, p. 27)

In depth study of videos, through the establishment and use of new collaborators, suggests a new direction for research and practice in education. These collaborators have the potential of building new communities to share important video collections that support both basic research and professional development. They offer invitations for researchers and practitioners to interact and collaborate on research. They offer new ways to disseminate research and invite multidisciplinary perspectives from a variety of fields, including the learning sciences, developmental psychology, and mathematics education. Video-based technological advances hold great promise for the future.

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